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Completely Indecomposable Operators and a Uniqueness Theorem of Cartwright–Levinson Type

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A bounded linear operator T on a complex Hilbert space will be called completely indecomposable if its spectrum is not a singleton, and is included in the spectrum of the restrictions of T and T^* to any of their nonzero invariant sub-

second of bilateral weighted shifts whose spectrum is the unit circle. We do not know whether any of the operators in the first class has a proper invariant subspace and if any of the operators in the second class has a proper hyperinvariant subspace. We also establish a new uniqueness theorem of Cartwright–Levinson type which is the main ingredient in our proofs of complete indecomposability. © 1999 Academic Press

1. INTRODUCTION

Let T be a bounded linear operator on a complex Hilbert space H. We shall say that it is *strongly indecomposable*, if its spectrum $\sigma(T)$ is not a singleton, and is included in the spectrum of the restriction of T to any of its nonzero invariant subspaces. If both T and its adjoint T^* are strongly indecomposable, we shall say that T is *completely indecomposable*.

It is easy to find examples of strongly indecomposable operators. If the spectrum of T is not a singleton and has a dense subset E such that

$$\bigcap_{n=1}^{\infty} (T-\lambda I)^n H=\{0\}, \qquad \forall \lambda \in E,$$

then T is strongly indecomposable, since this condition implies that for every $\lambda \in E$, the restriction of $T-\lambda I$ to any nonzero invariant subspace of T is not surjective.

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Thus for example, the unilateral shift S in $l^2(\mathbf{Z}_+)$ is strongly indecomposable, since the condition above holds for every λ in the open unit disc $\mathbf{D}$, and $\sigma(S) = \bar{\mathbf{D}}$. Another example of this type is an operator T with spectrum $[0, 1]$ which acts as multiplication by the identity function on a Hilbert space of C^∞-functions on $[0, 1]$ which belong to a quasianalytic class. Such an operator is strongly indecomposable, since for every $\lambda \in [0, 1]$, the space $\bigcap_{n=1}^{\infty} (T - \lambda I)^n H$ consists of functions in H which vanish at λ together with all their derivatives, and therefore by the assumption on H, this is the zero space. A concrete example of such an operator was given by Lyubich and Matsaev in [24].

The operators discussed above are not completely indecomposable since their adjoints have eigenvalues.

In this paper, we construct two classes of completely indecomposable operators. The first class consists of operators with spectrum $[-2, 2]$, which are perturbations of a self-adjoint operator by a compact quasinilpotent operator. We do not know whether any of the operators in this class has a proper invariant subspace. The second class consists of invertible bilateral weighted shifts whose spectrum is the unit circle, which are perturbations of the ordinary bilateral shift by a compact quasinilpotent operator. We do not know whether any of the operators in this class has a proper hyperinvariant subspace.

It seems that completely indecomposable operators have not been considered explicitly in the literature before. However, using a spectral mapping theorem of Fuhrmann [16], one can prove that the unicellular operators with non-singleton spectrum constructed by Foiaş and Williams in [15], are completely indecomposable. We thank Dr. D. Yakubovich for bringing this fact to our attention.

The main ingredient in the proofs that the operators mentioned above are completely indecomposable is a new uniqueness theorem of Cartwright–Levinson type, concerning analytic extension of holomorphic functions in the complement of the unit circle.

The rest of the paper is organized as follows. In Section 2 we state our main results, and in Section 3 we present their proofs. Section 4 contains additional results and open problems. In an Appendix, we give some examples.

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2. STATEMENT OF MAIN RESULTS

In what follows, $\mathbf{Z}$ denotes the set of all integers, and $\mathbf{Z}_+$ the set of all non-negative integers. We use the standard notations $\mathbf{T}$ and $\mathbf{D}$ for the unit circle and the open unit disc respectively.

If γ is a bounded sequence of real numbers on $\mathbf{Z}_+$, we denote by $S(\gamma)$ the unilateral weighted shift on $l^2(\mathbf{Z}_+)$ with weight sequence γ, that is the bounded linear operator on this space defined by

$$S(\gamma)\, e_n = \gamma(n)\, e_{n+1}, \qquad n \in \mathbf{Z}_+,$$

where $\{e_n\}_{n \in \mathbf{Z}_+}$ is the standard orthonormal basis of $l^2(\mathbf{Z}_+)$.

Throughout this paper, we assume that α is a sequence of positive numbers on $\mathbf{Z}_+$ such that

$$0 < \inf_{n \in \mathbf{Z}_+} \alpha(n) \leqslant \sup_{n \in \mathbf{Z}_+} \alpha(n) < \infty. \tag{2.1}$$

The sequence $\{\alpha(n)^{-1}\}_{n \in \mathbf{Z}_+}$ will be denoted by α^{-1}.

The operator $A(\alpha) = S(\alpha) + S^*(\alpha^{-1})$ will be called the *weighted bi-shift* with weight sequence α. In the orthonormal basis $\{e_n\}$ it has a three-diagonal matrix representation:

$$A(\alpha) = \begin{pmatrix} 0 & \alpha^{-1}(0) & 0 & \cdots & \cdots \\ \alpha(0) & 0 & \alpha^{-1}(1) & 0 & \cdots \\ 0 & \alpha(1) & 0 & \alpha^{-1}(2) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We describe now a one-to-one correspondence between the class of weighted bi-shifts and a class of operators on certain Hilbert spaces of holomorphic functions. Assume that ω is a sequence of positive numbers on $\mathbf{Z}_+$ such that

$$0 < \inf_{n \in \mathbf{Z}_+} \frac{\omega(n+1)}{\omega(n)} \leqslant \sup_{n \in \mathbf{Z}_+} \frac{\omega(n+1)}{\omega(n)} < \infty, \tag{2.2}$$

and denote $R(\omega) = \liminf_{n \to \infty} [\omega(n)]^{1/n}$. It follows from (2.2) that $0 < R(\omega) < \infty$. Let $H^2(\omega)$ denote the vector space of all holomorphic functions f on the disc $D_\omega = \{z \in \mathbf{C}\colon |z| < R(\omega)\}$, such that the norm

$$\|f\|_\omega = \left(\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right|^2 \omega^2(n) \right)^{1/2}$$

is finite. It is easily verified that, with respect to this norm, $H^2(\omega)$ is a Hilbert space, and the linear operator B_ω defined by

$$(B_\omega f)(z) = \begin{cases} zf(z) + z^{-1}[f(z) - f(0)] & \text{if } \ z \in D_\omega \setminus \{0\} \\ f'(0), & \text{if } \ z = 0 \end{cases}$$

is bounded on this space. Consider the sequence α on $\mathbf{Z}_+$ defined by

$$\alpha(n) = \frac{\omega(n+1)}{\omega(n)}, \qquad n \in \mathbf{Z}_+.$$

It follows from (2.2) that α satisfies (2.1). The operator B_ω is unitary equivalent to $A(\alpha)$. In fact, if $V_\omega\colon H^2(\omega) \to l^2(\mathbf{Z}_+)$ is the unitary operator defined by

$$V_\omega f = \left\{ \frac{f^{(n)}(0)}{n!}\, \omega(n) \right\}_{n \in \mathbf{Z}_+}, \qquad f \in H^2(\omega),$$

then $B_\omega = V_\omega^{-1} A(\alpha)\, V_\omega$.

Conversely, if α is a sequence of positive numbers on $\mathbf{Z}_+$ which satisfies (2.1), then the sequence ω_α on $\mathbf{Z}_+$ defined by

$$\omega_\alpha(n) = \begin{cases} \prod_{j=0}^{n-1} \alpha(j), & \text{for} \quad n > 0 \\ 1, & \text{for} \quad n = 0 \end{cases} \tag{2.3}$$

satisfies condition (2.2) and $A(\alpha) = V_\omega B_\omega V_\omega^{-1}$ with $\omega = \omega_\alpha$.

The adjoint of a weighted bi-shift is also an operator of this type since $A^*(\alpha) = A(\alpha^{-1})$, and the vector e_0 is cyclic for every such operator, since

$$\bigvee_{j=0}^{n} A^j(\alpha)\, e_0 = \bigvee_{j=0}^{n} e_j, \qquad n \in \mathbf{Z}_+.$$

We shall be primarily concerned with weighted bi-shifts $A(\alpha)$ such that

$$\lim_{n \to \infty} \alpha(n) = 1. \tag{2.4}$$

This condition implies that the operator $K(\alpha) = S(\alpha^{-1} - \alpha)$ is compact and quasinilpotent (cf. [26, Proposition 4, Theorem 4, and Proposition 15]), and therefore, since $A(\alpha) = S(\alpha) + S^*(\alpha) + K^*(\alpha)$, the operator $A(\alpha)$ is a perturbation of a self-adjoint operator by a compact quasinilpotent operator, hence, in particular, is essentially self-adjoint. A similar argument shows that the operator $A(\alpha) - (S + S^*)$ (where S is the unilateral shift) is also compact, but in general it is not quasinilpotent. As will be seen in the next section, condition (2.4) implies that $\sigma(A(\alpha)) = [-2, 2]$.

For a bounded linear operator T on a complex Hilbert space, we shall denote as usual by Lat T the collection of all invariant subspaces of T. For M in Lat T, we denote by $T|_M$ the restriction of T on M.

Our first result is

THEOREM 1. *Assume that* α *is a positive sequence on* $\mathbf{Z}_+$ *which satisfies* (2.4) *and also the conditions*

$$\left(\frac{\alpha(n-1)}{\alpha(n)}\right)^n = O(1), \qquad n \to \infty, \tag{2.5}$$

and

$$\limsup_{n\to\infty} \sum_{j=0}^{n} \frac{\log \omega_\alpha(j)}{j^2+1} = \infty, \tag{2.6}$$

where ω_α *is defined by* (2.3). *Then* $\sigma(A(\alpha)|_M) = [-2, 2]$ *for every* $M \neq \{0\}$ *in* Lat $A(\alpha)$; *hence* $A(\alpha)$ *is strongly indecomposable.*

If, in addition,

$$\left(\frac{\alpha(n+1)}{\alpha(n)}\right)^n = O(1), \qquad n \to \infty, \tag{2.7}$$

and

$$\liminf_{n\to\infty} \sum_{j=0}^{n} \frac{\log \omega_\alpha(j)}{j^2+1} = -\infty, \tag{2.8}$$

then $\sigma(A^*(\alpha)|_N) = [-2, 2]$ *for every* $N \neq \{0\}$ *in* Lat $A^*(\alpha)$, *hence* $A(\alpha)$ *is completely indecomposable.*

Since $A^*(\alpha) = A(\alpha^{-1})$, the second assertion of the theorem follows from the first one.

In view of (2.1), conditions (2.6) and (2.8) can be written directly in terms of α: they say that

$$\limsup_{n\to\infty} \sum_{j=1}^{n} \frac{\log \alpha(j)}{j} = \infty$$

and

$$\liminf_{n\to\infty} \sum_{j=1}^{n} \frac{\log \alpha(j)}{j} = -\infty,$$

respectively.

We turn to the second class of operators. If β is a bounded sequence of positive numbers on $\mathbf{Z}$, we denote by $U(\beta)$ the bilateral weighted shift on $l^2(\mathbf{Z})$, that is, the bounded linear operator on this space defined by

$$U(\beta)\, e_n = \beta(n)\, e_{n+1}, \qquad n \in \mathbf{Z},$$

where $\{e_n\}_{n \in \mathbf{Z}}$ is the standard orthonormal basis of $l^2(\mathbf{Z})$. The unweighted bilateral shift on $l^2(\mathbf{Z})$ will be denoted by U. We consider in the sequel mainly bilateral weighted shifts with weight sequence β satisfying the condition

$$\lim_{n \to \pm\infty} \beta(n) = 1. \tag{2.9}$$

This implies, as in the case of bi-shifts, that the operator $U(\beta) - U$ is compact and quasinilpotent, and by [26, Proposition 15 and Corollary of Theorem 7], that $\sigma(U(\beta)) = \mathbf{T}$.

For every sequence of positive numbers β on $\mathbf{Z}$, we denote by W_β the sequence on $\mathbf{Z}$ defined by

$$W_\beta(n) = \begin{cases} \prod_{j=0}^{n-1} \beta(j) & n > 0 \\ 1 & n = 0 \\ \prod_{j=n}^{-1} \beta^{-1}(j) & n < 0. \end{cases} \tag{2.10}$$

THEOREM 2. *Assume that β is a positive sequence on $\mathbf{Z}$ which satisfies (2.9) and also the conditions*

$$\left(\frac{\beta(n-1)}{\beta(n)}\right)^{|n|} = O(1), \qquad n \to \pm\infty \tag{2.11}$$

and

$$\limsup_{n \to \infty} \sum_{j=0}^{n} \frac{\log[W_\beta(j)\, W_\beta(-j)]}{j^2 + 1} = \infty. \tag{2.12}$$

Then

$$\sigma(U(\beta)|_M) = \mathbf{T}, \qquad \forall M \neq \{0\} \text{ in } \operatorname{Lat} U(\beta) \cap \operatorname{Lat} U^{-1}(\beta)$$

and

$$\sigma(U(\beta)|_M) = \overline{\mathbf{D}}, \qquad \forall M \neq \{0\} \text{ in } \operatorname{Lat} U(\beta) \backslash \operatorname{Lat} U^{-1}(\beta).$$

Hence, in particular, $U(\beta)$ is strongly indecomposable.

If, in addition,

$$\liminf_{n\to\infty} \sum_{j=0}^{n} \frac{\log[W_\beta(j)\, W_\beta(-j)]}{j^2+1} = -\infty, \tag{2.13}$$

then

$$\sigma(U^*(\beta)|_N) = \mathbf{T}, \qquad \forall N \neq \{0\} \textit{ in } \operatorname{Lat} U^*(\beta) \cap \operatorname{Lat} U^{*-1}(\beta)$$

and

$$\sigma(U^*(\beta)|_N) = \bar{\mathbf{D}}, \qquad \forall N \neq \{0\} \textit{ in } \operatorname{Lat} U^*(\beta) \backslash \operatorname{Lat} U^{*-1}(\beta).$$

Thus, in particular, $U(\beta)$ is completely indecomposable.

The second assertion of the theorem follows from the first one, since $U^*(\beta)$ is unitarily equivalent to the operator $U(\delta)$ where $\delta(n) = \beta(-n-1)$, $n \in \mathbf{Z}$, and therefore $W_\delta(n) = W_\beta^{-1}(-n)$, $n \in \mathbf{Z}$.

We recall that a hyperinvariant subspace for an operator T on a Hilbert space, is a subspace which is invariant under all operators that commute with T. It is known [26, Corollary of Theorem 12], that the hyperinvariant subspaces of an invertible bilateral weighted shift $U(\beta)$ are precisely the subspaces in $\operatorname{Lat} U(\beta) \cap \operatorname{Lat} U^{-1}(\beta)$. Hence Theorem 2 implies that if β satisfies conditions (2.9)–(2.13), and M and N are nonzero hyperinvariant subspaces of $U(\beta)$ and $U^*(\beta)$ respectively, then $\sigma(U(\beta)|_M) = \sigma(U^*(\beta)|_N) = \mathbf{T}$.

Examples of sequences α and β which satisfy the conditions of Theorems 1 and 2 will be given in the Appendix.

Before stating the uniqueness theorem mentioned in the Introduction, we describe some results in this area, in order to place it in proper perspective. For this, we need some definitions and notations.

A formal trigonometric series

$$\Phi \sim \sum_{n\in\mathbf{Z}} \hat{\Phi}(n)\, e^{in\theta}$$

with complex coefficients $\hat{\Phi}(n)$ which for every $\varepsilon > 0$ satisfy the condition

$$|\hat{\Phi}(n)| = O(e^{\varepsilon |n|}), \qquad n \to \pm\infty,$$

is called a *hyperdistribution on* $\mathbf{T}$ (cf. [18, Appendix I]). The vector space of all such hyperdistributions will be denoted by $\mathscr{H}(\mathbf{T})$. A function in $L^1(\mathbf{T})$ is identified with the hyperdistribution determined by its Fourier series. If Φ is in $\mathscr{H}(\mathbf{T})$ we define its Cauchy transform $\mathscr{C}(\Phi)$ to be the holomorphic function on $\hat{\mathbf{C}}\backslash\mathbf{T}$ given by

$$\mathscr{C}(\Phi)(z)=\begin{cases}\sum_{n=0}^{\infty}\hat{\Phi}(n)\,z^n, & z\in\mathbf{D}\\ -\sum_{n=-\infty}^{-1}\hat{\Phi}(n)\,z^n, & z\in\hat{\mathbf{C}}\backslash\bar{\mathbf{D}}.\end{cases}$$

A simple computation shows that if Φ is a function in $L^1(\mathbf{T})$, then

$$\mathscr{C}(\Phi)(z)=\frac{1}{2\pi i}\int_{\mathbf{T}}\frac{\Phi(\zeta)}{\zeta-z}\,d\zeta,\qquad z\in\hat{\mathbf{C}}\backslash\mathbf{T}.$$

It is clear that the mapping $\Phi\mapsto\mathscr{C}(\Phi)$ is an isomorphism of the vector space $\mathscr{H}(\mathbf{T})$ onto the vector space of all holomorphic functions on $\hat{\mathbf{C}}\backslash\mathbf{T}$ which vanish at infinity.

One says that a hyperdistribution Φ in $\mathscr{H}(\mathbf{T})$ vanishes on an open subarc J of $\mathbf{T}$, if the function $\mathscr{C}(\Phi)$ has a holomorphic extension to the domain $(\hat{\mathbf{C}}\backslash\mathbf{T})\cup J$. It is easy to show that if Φ is a function in $L^1(\mathbf{T})$, then it vanishes on the open arc J as a hyperdistribution, if and only if, $\Phi=0$ a.e. on J (cf. [18, Appendix I]).

A non-zero subspace of $\mathscr{H}(\mathbf{T})$ will be called *quasianalytic*, if it contains no non-zero elements which vanish on some open subarc of $\mathbf{T}$.

If p is a sequence of real numbers on $\mathbf{Z}$ such that

$$\liminf_{n\to\pm\infty}\frac{p(n)}{|n|}\geqslant 0,$$

we shall denote by $\mathscr{H}_p(\mathbf{T})$ the vector space of all formal trigonometric series Φ such that

$$|\hat{\Phi}(n)|=O(e^{-p(n)}),\qquad n\to\pm\infty.$$

The assumption on p implies that $\mathscr{H}_p(\mathbf{T})\subset\mathscr{H}(\mathbf{T})$.

The classical Denjoy–Carleman theorem on quasi-analytic classes of C^∞ functions on $\mathbf{T}$ (cf. [19, p. 114]) is equivalent to the assertion that if p is an even sequence of positive numbers on $\mathbf{Z}$ such that the sequence $p|_{\mathbf{Z}_+}$ is concave (that is $p(n+1)+p(n-1)-2p(n)\leqslant 0$, $n=1, 2, ...$) and

$$\lim_{n\to\infty}\frac{p(n)}{\log n}=\infty, \tag{2.14}$$

then the space $\mathscr{H}_p(\mathbf{T})$ is quasianalytic if and only if

$$\sum_{n\in\mathbf{Z}} \frac{p(n)}{n^2+1} = \infty. \tag{2.15}$$

Note that condition (2.14) implies that $\mathscr{H}_p(\mathbf{T}) \subset C^\infty(\mathbf{T})$.

This is not the standard statement of the Denjoy–Carleman theorem, but can be shown to be equivalent to it, and also follows from a more general result of Domar [10, Theorem 2.11] (see also [6, Sect. 1]). A recent result of Koosis [20, Corollary of Theorem 4] implies that if p is an arbitrary positive sequence on $\mathbf{Z}$, then condition (2.15) is necessary for the quasianalyticity of the space $\mathscr{H}_p(\mathbf{T})$.

Cartwright and Levinson made a significant extension of the sufficiency part of the Denjoy–Carleman theorem. They proved (cf. [9] and [23, Theorem XVI]) that if q is a non-negative concave sequence on $\mathbf{Z}_+$ such that

$$\sum_{n=0}^{\infty} \frac{q(n)}{n^2+1} < \infty, \tag{2.16}$$

and θ is a non-negative increasing sequence on $\mathbf{Z}_+$ such that

$$\sum_{n=0}^{\infty} \frac{\theta(n)}{n^2+1} = \infty, \tag{2.17}$$

then the space $\mathscr{H}_p(\mathbf{T})$ with

$$p(n) = \begin{cases} -q(n), & n \geqslant 0 \\ \theta(-n), & n < 0 \end{cases}$$

is quasianalytic. They formulated their result in different form, but one can show that it is equivalent to the statement above by using a result of Beurling on Legendre transforms [5, Lemma 1].

Our uniqueness result is

THEOREM 3. *If p is a sequence of real numbers on $\mathbf{Z}$ which satisfies the conditions*

$$p(n+1) - p(n) = o(1), \qquad n \to \pm\infty, \tag{2.18}$$

$$\sup_{n\in\mathbf{Z}} |n| \, (2p(n) - m(n+1) - p(n-1)) < \infty, \tag{2.19}$$

and

$$\limsup_{n \to \infty} \sum_{j=0}^{n} \frac{p(j)+p(-j)}{j^2+1} = \infty, \tag{2.20}$$

then the space $\mathscr{H}_p(\mathbf{T})$ *is quasianalytic.*

Condition (2.18) implies that

$$p(n) = o(n), \qquad n \to \pm\infty,$$

and therefore $\mathscr{H}_p(\mathbf{T}) \subset \mathscr{H}(\mathbf{T})$. It is worth noting that condition (2.19) is merely a regularity condition, and does not impose further growth restriction on the sequence p other than the one above. For example, when (2.18) holds, then condition (2.19) is satisfied whenever the sequences $\{p(n+1)-p(n)\}_{n \in \mathbf{Z}_+}$ and $\{p(-n+1)-p(-n)\}_{n \in \mathbf{Z}_+}$ are concave or convex. In fact, in this case (2.18) implies that

$$\lim_{n \to \pm\infty} n(2p(n) - p(n+1) - p(n-1)) = 0.$$

It is clear that for sequences which satisfy conditions (2.18) and (2.19), Theorem 3 extends the result of Cartwright and Levinson. Condition (2.20) is considerably more flexible than conditions (2.16) and (2.17). Whereas these conditions require that the coefficients of an element in $\mathscr{H}_p(\mathbf{T})$ have moderate growth as $n \to +\infty$ and rapid decay as $n \to -\infty$, no such one sided conditions are imposed by (2.20).

The major difference between Theorem 3 and other results of Cartwright–Levinson type (cf. Beurling [4, pp. 396–431], Borichev and Volberg [6]) is that it also provides sequences p such that both of the spaces $\mathscr{H}_p(\mathbf{T})$ and $\mathscr{H}_{-p}(\mathbf{T})$ are quasianalytic. This is crucial for our construction of completely indecomposable operators. On the other hand, Beurling, Borichev, and Volberg deduce from their assumptions considerably stronger forms of quasianalyticity than the one defined above.

3. PROOFS OF MAIN RESULTS

We first prove Theorems 1 and 2 by using Theorem 3, which will be proved in the end of this section.

In what follows, $\mathscr{L}(H)$ denotes the algebra of all bounded linear operators on a complex Hilbert space H. For an operator T in $\mathscr{L}(H)$ we denote by $\rho(T)$ its resolvent set, by $\rho_\infty(T)$ the set $\rho(T) \cup \{\infty\}$, and by $R_T(\cdot)$ the resolvent of T, which we regard here as the function $\zeta \mapsto (\zeta I - T)^{-1}$ on $\rho_\infty(T)$ vanishing at infinity.

We begin by showing that condition (2.4) implies that $\sigma(A(\alpha)) = [-2, 2]$. The function $\zeta \mapsto \zeta + \zeta^{-1}$ on $\hat{\mathbf{C}}$ will be denoted in the sequel by ψ.

LEMMA 3.1. *Assume that T_1 and T_2 are operators in $\mathscr{L}(H)$ such that $T_2 T_1 = I$, and let $T = T_1 + T_2$. Then*

$$\psi(\rho(T_1) \cap \rho(T_2)) \subset \rho_\infty(T),$$

and

$$R_T(\psi(\zeta)) = \zeta R_{T_1}(\zeta)\, R_{T_2}(\zeta), \qquad \forall \zeta \in \rho(T_1) \cap \rho(T_2).$$

Proof. It follows from the assumption that

$$\psi(\zeta)\, I - T = \zeta^{-1}(\zeta I - T_2)(\zeta I - T_1), \qquad \forall \zeta \in \mathbf{C} \backslash \{0\},$$

and this implies both assertions. ∎

COROLLARY 3.2. *If T_1, T_2 and T are as in Lemma* 3.1, *and $\sigma(T_1) \cup \sigma(T_2) \subset \bar{\mathbf{D}}$, then $\sigma(T) \subset [-2.2]$.*

Proof. The hypothesis implies that $\mathbf{C} \backslash \bar{\mathbf{D}} \subset \rho(T_1) \cap \rho(T_2)$, and therefore, since $\psi(\mathbf{C} \backslash \bar{\mathbf{D}}) = \mathbf{C} \backslash [-2, 2]$, the conclusion follows from Lemma 3.1. ∎

PROPOSITION 3.3. *If α is a sequence of positive numbers on $\mathbf{Z}_+$ which satisfies condition* (2.4), *then $\sigma(A(\alpha)) = [-2, 2]$.*

Proof. Condition (2.4) implies that $\sigma(S(\alpha)) = \sigma(S^*(\alpha^{-1})) = \bar{\mathbf{D}}$ [26, Theorem 4 and Proposition 15], and therefore, since $S^*(\alpha^{-1}) S(\alpha) = I$, it follows from Corollary 3.2 that $\sigma(A(\alpha)) \subset [-2, 2]$. By Lemma 3.1,

$$R_{A(\alpha)}(\psi(\zeta)) = \zeta R_{S(\alpha)}(\zeta)\, R_{S^*(\alpha^{-1})}(\zeta), \qquad \forall \zeta \in \mathbf{C} \backslash \bar{\mathbf{D}}.$$

Since $\psi(\zeta) = \psi(\zeta^{-1})$, this implies that

$$R_{A(\alpha)}(\psi(\zeta)) = \zeta (I - \zeta S(\alpha))^{-1} (I - \zeta S^*(\alpha^{-1}))^{-1}, \qquad \forall \zeta \in \mathbf{D}. \tag{3.1}$$

For x in $l^2(\mathbf{Z}_+)$, let f_x denote the holomorphic function on $\hat{\mathbf{C}} \backslash \mathbf{T}$ defined by

$$f_x(\zeta) = \begin{cases} \sum_{n=0}^{\infty} x(n)\, \omega_\alpha^{-1}(n)\, \zeta^{n+1}, & \zeta \in \mathbf{D} \\ \sum_{n=-\infty}^{0} x(|n|)\, \omega_\alpha^{-1}(|n|)\, \zeta^{n-1}, & \zeta \in \hat{\mathbf{C}} \backslash \bar{\mathbf{D}} \end{cases}$$

where ω_α is defined by (2.3). Observing that for $\zeta \in \mathbf{D}$,

$$(I-\bar{\zeta}S^*(\alpha))^{-1}\,e_0=\sum_{n=0}^{\infty}\bar{\zeta}^n(S^*(\alpha))^n\,e_0=e_0,$$

and

$$(I-\bar{\zeta}S(\alpha^{-1}))^{-1}\,e_0=\sum_{n=0}^{\infty}\frac{\bar{\zeta}^n}{\omega_\alpha(n)}\,e_n,$$

we obtain from (3.1) (by using again the fact that $\psi(\zeta)=\psi(\zeta^{-1})$) that

$$\langle R_{A(\alpha)}(\psi(\zeta))\,x, e_0\rangle = f_x(\zeta), \qquad \forall \zeta \in \hat{\mathbf{C}}\backslash T. \tag{3.2}$$

Setting $x=e_0$, we see that the function

$$\zeta \mapsto \langle R_{A(\alpha)}(\psi(\zeta))\,e_0, e_0\rangle, \qquad \zeta \in \hat{\mathbf{C}}\backslash\mathbf{T}$$

assumes the value ζ for $\zeta \in \mathbf{D}$, and the value ζ^{-1} for $\zeta \in \hat{\mathbf{C}}\backslash\bar{\mathbf{D}}$, and therefore it has no analytic extension to any open set which contains a point of $\mathbf{T}$. Since ψ is holomorphic in $\mathbf{C}\backslash\{0\}$ and $\psi(\mathbf{T})=[-2,2]$, it follows that the function $R_{(\alpha)}(\cdot)$ has no analytic extension to any open set which contains a point of $[-2,2]$. Thus $[-2,2]\subset\sigma(A(\alpha))$. Since we already know that the inverse inclusion also holds, the proof is complete. ∎

To proceed we need an additional notation. If K is a compact set in $\mathbf{C}$, we shall denote by K^f the *filling* of K; i.e. the complement of the unbounded component of $\mathbf{C}\backslash K$, or equivalently, the union of K and the bounded components of $\mathbf{C}\backslash K$. The set K^f is also called the polynomially convex hull of K. In the sequel, we shall use the following

LEMMA 3.4 [25, Theorem 0.8]. *If T is in $\mathscr{L}(H)$ and M is in* Lat T, *then* $\sigma(T|_M)\subset\sigma^f(T)$.

It follows from this result and Proposition 3.3, that if α satisfies condition (2.4), then $\sigma(A(\alpha)|_M)\subset[-2,2]$, for every $M\in \operatorname{Lat} A(\alpha)$.

Before turning to the proof of Theorem 1, we recall some definitions from local spectral theory (cf. [12, p. 1931]), and make some observations. If T is in $\mathscr{L}(H)$ and x is in H, then a holomorphic H-valued function f on some open set $D(f)$ in $\mathbf{C}$, is called an *analytic extension* of $R_T(\cdot)\,x$, if $\rho(T)\subset D(f)$ and

$$(\lambda I-T)\,f(\lambda)=x, \qquad \forall \lambda\in D(f).$$

It is clear that such an analytic extension coincides with $R_T(\cdot)\,x$ on $\rho(T)$. The complement of the union of all open sets $D(f)$ for all analytic extensions f of $R_T(\cdot)\,x$, is called the local spectrum of x (with respect to T), and is denoted by $\sigma(x, T)$.

It follows from these definitions that if M is in Lat T, then $\sigma(x, T)$ is a closed subset of $\sigma(T|_M)$ for every $x\in M$; in particular, $\sigma(x, T)\subset\sigma(T)$ for every $x \in H$. Hence, if $\sigma(T)$ is not a singleton and $\sigma(x, T) = \sigma(T)$ for every $x \neq 0$ in H, then T is strongly indecomposable.

It is easy to see that if $\sigma(T)$ has empty interior and $x \in H$, then $\sigma(x, T) = \sigma(T)$, if and only if, there is no holomorphic H-valued function on an open set in $\mathbf{C}$ which properly includes $\rho(T)$, and coincides with $R_T(\cdot)\, x$ on $\rho(T)$.

Proof of Theorem 1. By Proposition 3.3, Lemma 3.4 and the preceding observations, it suffices to show that $\sigma(x, A(\alpha)) = [-2, 2]$ for every $x \neq 0$ in $l^2(\mathbf{Z}_+)$. This will clearly follow if we show that the function

$$\lambda \mapsto \langle R_{A(\alpha)}(\lambda)\, x, e_0 \rangle, \qquad \lambda \in \hat{\mathbf{C}} \backslash [-2, 2]$$

has no analytic extension to any open set which contains a point of $[-2, 2]$. Since ψ is holomorphic in $\mathbf{C} \backslash \{0\}$, and $\psi(\mathbf{T}) = [-2, 2]$, it suffices to show that the function

$$\zeta \mapsto \langle R_{A(\alpha)}(\psi(\zeta))\, x, e_0 \rangle, \qquad \zeta \in \hat{\mathbf{C}} \backslash \mathbf{T},$$

has no holomorphic extension to any open set which contains a point of T. It follows from (3.2) that this function is the Cauchy transform of the hyperdistribution

$$\Phi \sim \sum_{j \in \mathbf{Z} \backslash \{0\}} \operatorname{sign}(j)\, \frac{x(|j|-1)}{\omega_\alpha(|j|-1)}\, e^{ij\theta}.$$

Consider the sequence

$$p(n) = \log \omega_\alpha(|n|), \qquad n \in \mathbf{Z}.$$

Since α is bounded, Φ is in $\mathscr{H}_p(\mathbf{T})$. Assumptions (2.4)–(2.6) imply that p satisfies the conditions of Theorem 3, and therefore the space $\mathscr{H}_p(\mathbf{T})$ is quasi-analytic. Thus, the Cauchy transform of Φ has no holomorphic extension to an open set which contains a point of $\mathbf{T}$, and the proof is complete. ∎

For the proof of Theorem 2 we need a preliminary result.

PROPOSITION 3.5. *Let Ω be a bounded domain in $\mathbf{C}$ which contains the origin, such that $\mathbf{C} \backslash \bar{\Omega}$ is connected, and let $\Gamma = \partial\Omega$. Assume that T is an operator in $\mathscr{L}(H)$ such that $\sigma(T) \subset \Gamma$. Then $\sigma(T|_M) \subset \Gamma$ if M is in* Lat $T \cap$ Lat T^{-1}, *and $\sigma(T|_M) = \bar{\Omega}$ if $M \in$* Lat $T \backslash$Lat T^{-1}.

Proof. The assumptions on Ω imply that $\Gamma^f = \bar{\Omega}$, and since $\sigma(T) \subset \Gamma$, it follows from Lemma 3.4 that $\sigma(T|_M) \subset \bar{\Omega}$ for every M in Lat T. Assume that $M \in \text{Lat}\, T \cap \text{Lat}\, T^{-1}$, and denote the function $z \mapsto z^{-1}$ on $\hat{\mathbf{C}}$ by g. Using again the assumptions on Ω, we get that $[g(\Gamma)]^f = g(\hat{\mathbf{C}} \backslash \Omega)$, and therefore, since $\sigma(T^{-1}) = g(\sigma(T)) \subset g(\Gamma)$, and $(T|_M)^{-1} = T^{-1}|_M$, we obtain from Lemma 3.4 that $\sigma(T|_M^{-1}) \subseteq g(\hat{\mathbf{C}} \backslash \Omega)$, and consequently,

$$\sigma(T|_M) = g(\sigma(T|_M^{-1})) \subseteq \hat{\mathbf{C}} \backslash \Omega.$$

Remembering that $\sigma(T|_M) \subset \bar{\Omega}$, we conclude that $\sigma(T|_M) \subset \Gamma$.

Assume now that $M \in \text{Lat}\, T \backslash \text{Lat}\, T^{-1}$, and consider the sets $G = \rho(T|_M) \cap \Omega$ and $F = \{\lambda \in \rho(T) : (\lambda I - T)M = M\}$. Since $\Omega \subset \rho(T)$, it follows that $G = F \cap \Omega$, and therefore, since F is closed in $\rho(T)$, we obtain that G is open and closed in Ω. Since Ω is connected, we conclude that either $G = \varnothing$ or $G = \Omega$. But the assumption that M does not belong to Lat T^{-1} implies that $T|_M$ is not surjective, so that $0 \in \Omega \backslash \rho(T|_M)$, and the second option is ruled out. This shows that $\bar{\Omega} \subset \sigma(T|_M)$, and as we already have the opposite inclusion, the proof is complete. ▮

Proof of Theorem 2. As already observed in Section 2, condition (2.9) implies that $\sigma(U(\beta)) = \mathbf{T}$. Hence by Proposition 3.5 and the preceding observations concerning the local spectrum, the theorem will be proved if we show that

$$\sigma(x, U(\beta)) = \mathbf{T}, \qquad \forall x \neq \{0\} \text{ in } l^2(\mathbf{Z}).$$

To show this, consider the formal trigonometric series

$$\Phi \sim \sum_{n \in \mathbf{Z}} \frac{x(n)}{W_\beta(n)} e^{in\theta}$$

and the sequence

$$p(n) = \log W_\beta(n), \qquad n \in \mathbf{Z},$$

where W_β is defined by (2.10). It is clear that Φ is in $\mathscr{H}_p(\mathbf{T})$, and assumptions (2.9), (2.11) and (2.12) imply that p satisfies the conditions of Theorem 3, hence the space $\mathscr{H}_p(\mathbf{T})$ is quasianalytic. Using the expansion of the resolvent of $U(\beta)$ in $\hat{\mathbf{C}} \backslash \mathbf{T}$, we obtain that

$$\mathscr{C}(\Phi)(\zeta) = \langle (I - \zeta R_{U(\beta)}(\zeta))\, x, e_0 \rangle, \qquad \forall \zeta \in \hat{\mathbf{C}} \backslash \mathbf{T},$$

and since the space $\mathscr{H}_p(\mathbf{T})$ is quasianalytic, this implies that the vector function

$$\zeta \mapsto R_{U(\beta)}(\zeta)\, x, \qquad \zeta \in \hat{\mathbf{C}} \backslash \mathbf{T}$$

has no analytic extension to an open set which contains a point of $\mathbf{T}$. Thus $\sigma(x, U(\beta)) = \mathbf{T}$, and the proof is complete. ▮

The proof of Theorem 3 is based on three results on entire functions which we state in a form suitable to our purpose.

For $\tau \geqslant 0$, we denote by $\mathscr{E}_\tau$ the vector space of all entire functions F which satisfy the condition

$$\sup_{z \in \mathbf{C}} |F(z)|\, e^{-(\tau+\varepsilon)\,|z|} < \infty, \qquad \forall \varepsilon > 0,$$

and by $\mathscr{E}_\tau^0$ the subspace of all functions F in $\mathscr{E}_\tau$ such that

$$\limsup_{r \to \infty} \frac{\log |F(re^{i\theta})|}{r} \leqslant \tau\, |\sin \theta|, \qquad \forall \theta \in [-\pi, \pi].$$

Thus $\mathscr{E}_\tau$ consists of all entire functions of exponential type at most τ, and $\mathscr{E}_\tau^0$ consists of all functions in $\mathscr{E}_\tau$ whose indicator diagram is included in the segment $\{z \in \mathbf{C} : |\Im z| \leqslant \pi\}$.

The first result is a special case of a theorem of F. Carlson which he proved in his thesis [7]. A more accessible reference is [22, Sect. 10.2].

PROPOSITION 3.6. *If $0 \leqslant \tau < \pi$ and Φ is an element of $\mathscr{H}(\mathbf{T})$ which vanishes on the arc $\{z \in \mathbf{T} \colon \tau < |\arg z| \leqslant \pi\}$, then there exists a function F in $\mathscr{E}_\tau^0$ such that $F(n) = \hat{\Phi}(n)$, $\forall n \in \mathbf{Z}$.*

The second result is a special case of a theorem of Agmon [1, Theorem 3a].

PROPOSITION 3.7. *Assume that p is a sequence on Z which satisfies conditions (2.18) and (2.19), and denote by p_* the continuous extension of p to $\mathbf{R}$ which is linear on the intervals $[n, n+1]$, $n \in \mathbf{Z}$. If $0 \leqslant \tau < \pi$, and F is a function in $\mathscr{E}_\tau^0$ such that*

$$\sup_{n \in \mathbf{Z}} |F(n)|\, e^{p(n)} < \infty,$$

then

$$\sup_{x \in \mathbf{R}} |F(x)|\, e^{p_*(x)} < \infty.$$

In the particular case where $p=0$, this was proved by Cartwright [8]. The third result is a uniqueness theorem of B. Levin.

PROPOSITION 3.8. *If $\tau \geqslant 0$, and F is a function in $\mathscr{E}_\tau$ which satisfies the condition*

$$\liminf_{R\to\infty} \int_0^R \frac{\log |F(x)\, F(-x)|}{x^2+1}\, dx = -\infty,$$

then $F \equiv 0$.

Levin obtains this result in the proof of Theorem 3 in Chapter V of [21] by using the Carleman formula; however, he does not formulate the result explicitly.

Proof of Theorem 3. If Φ is in $\mathscr{H}_p(\mathbf{T})$, then for every $\zeta \in \mathbf{T}$, the hyperdistribution

$$\Phi_\zeta \sim \sum_{n\in\mathbf{Z}} \hat{\Phi}(n)\, \zeta^n e^{in\theta}$$

is also in $\mathscr{H}_p(\mathbf{T})$, and

$$\mathscr{C}(\Phi_\zeta)(z) = \mathscr{C}(\Phi)(\zeta z), \qquad \forall z \in \hat{\mathbf{C}} \backslash \mathbf{T}.$$

Therefore, in order to prove that the space $\mathscr{H}_p(\mathbf{T})$ is quasianalytic, it suffices to show that, if $0 \leqslant \tau < \pi$ and Φ is an element in $\mathscr{H}_p(\mathbf{T})$ that vanishes on the arc $\{z \in \mathbf{T} : \tau < |\arg z| \leqslant \pi\}$, then $\Phi = 0$. But by Proposition 3.6, this assumption implies that there exists an entire function F in $\mathscr{E}_\tau$ such that $F(n) = \hat{\Phi}(n)$, $\forall n \in \mathbf{Z}$, and by Proposition 3.7, there exists a positive constant C such that

$$\int_0^n \frac{\log |F(x)\, F(-x)|}{x^2+1}\, dx \leqslant -\int_0^n \frac{p_*(x)+p_*(-x)}{x^2+1}\, dx + C, \qquad \forall n \in \mathbf{Z}_+. \tag{3.3}$$

A simple estimate, which uses (2.18), shows that

$$\int_0^n \frac{p_*(x)+p_*(-x)}{x^2+1}\, dx = \sum_{j=0}^{n} \frac{p(j)+p(-j)}{j^2+1} + O(1), \qquad n\to\infty.$$

This, together with (3.3) and (2.20), implies that

$$\liminf_{R\to\infty} \int_0^R \frac{\log |F(x)\, F(-x)|}{x^2+1}\, dx = -\infty.$$

Thus by Proposition 3.8, $F \equiv 0$, and therefore $\Phi = 0$. ∎

4. ADDITIONAL RESULTS AND PROBLEMS

It seems that the existing methods for producing proper invariant subspaces do not yield solutions to the following

Problem 1. Does every weighted bi-shift have a proper invariant subspace?

Problem 2. Does any of the weighted bi-shifts with weight sequence satisfying conditions (2.4)–(2.8) have a proper invariant subspace?

If α is a weight sequence such that either ω_α or ω_α^{-1} is in $l^2(\mathbf{Z}_+)$, then $A(\alpha)$ has proper invariant subspaces, since, if $0 < \theta < 2\pi$, $\theta \neq \pi$, then in the first case,

$$x_\theta = \{\omega_\alpha(n) \sin(n+1)\,\theta\}_{n \in \mathbf{Z}_+}$$

is an eigenvector of $A(\alpha)$ with eigenvalue $2\cos\theta$; and in the second case

$$y_\theta = \{\omega_\alpha^{-1}(n) \sin(n+1)\,\theta\}_{n \in \mathbf{Z}_+}$$

is an eigenvector of $A^*(\alpha)$ with the same eigenvalue.

It also follows from this observation and Theorem 1, that if α satisfies conditions (2.4) and (2.5), if ω_α^{-1} is in $l^2(\mathbf{Z}_+)$, and

$$\sum_{n=0}^{\infty} \frac{\log \omega_\alpha(n)}{n^2+1} = \infty,$$

then $A(\alpha)$ is an essentially self-adjoint strongly indecomposable operator with spectrum $[-2, 2]$, which is not completely indecomposable. A concrete example of this type is given by the sequence

$$\alpha(n) = \exp\left[\frac{n+2}{\log(n+3)} - \frac{n+1}{\log(n+2)}\right], \qquad n \in \mathbf{Z}_+.$$

Using the arguments in the proof of [2, Theorem 1.1], one can show that if α is a positive sequence on $\mathbf{Z}_+$ which satisfies condition (2.1) and

$$\sum_{n=0}^{\infty} \frac{|\log \omega_\alpha(n)|}{n^2+1} < \infty, \tag{4.1}$$

then $A(\alpha)$ has a proper invariant subspace. Condition (4.1) holds, in particular, if

$$\sum_{n=1}^{\infty} \frac{|\alpha(n)-1|}{n} < \infty.$$

Thus we see that if α is a weight sequence such that ω_α decreases or increases sufficiently fast (so that ω_α or ω_α^{-1} are in $l^2(\mathbf{Z}_+)$), or is of moderate growth and decay (so that (4.1) holds), then $A(\alpha)$ has a proper invariant subspace. However, we are unable to produce a sequence α satisfying assumptions of Theorem 1, such that $A(\alpha)$ has a proper invariant subspace. Apparently, new methods are needed to deal with the case when ω_α oscilates in a way that conditions (2.6) and (2.8) hold simultaneously.

It is not known if every invertible bilateral weighted shift has a proper bi-invariant subspace; i.e., a common invariant subspace with its inverse. As mentioned before, an equivalent formulation is whether every invertible bilateral shift has a proper hyperinvariant subspace.

If β is a bounded sequence of positive numbers on $\mathbf{Z}$, and W_β is defined by (2.10), it is proved in [2, Theorem 5] that, if

$$\sum_{n\in\mathbf{Z}} \frac{|\log W_\beta(n)|}{n^2+1} < \infty,$$

then $U(\beta)$ has a proper hyperinvariant subspace. More recent results on the existence of hyperinvariant subspaces of invertible bilateral shifts, can be found in [3, 11, 13, 14]. None of these results covers operators satisfying conditions (2.9) and (2.11)–(2.13).

Problem 3. Does any of the operators $U(\beta)$, where β satisfies conditions (2.9) and (2.11)–(2.13) have a proper hyperinvariant subspace?

We show next, that there is a connection between the existence of proper invariant subspaces for weighted bi-shifts, and the existence of proper hyperinvariant subspaces for certain bilateral weighted shifts. This will be established by showing first that every weighted bi-shift is unitarily equivalent to a part (i.e., to a restriction to an invariant subspace) of an operator of the form $T+T^{-1}$ where T is an invertible bilateral weighted shift. To show this we need some notations.

In what follows, we assume that W is a sequence of positive numbers on $\mathbf{Z}$ such that $W(0)=1$, and

$$0<\inf_{n\in\mathbf{Z}}\frac{W(n+1)}{W(n)}\leqslant\sup_{n\in\mathbf{Z}}\frac{W(n+1)}{W(n)}<\infty, \tag{4.2}$$

and denote by $L^2(W)$ (as in [26]), the Hilbert space of all formal Laurent series $f(z)=\sum_{n\in\mathbf{Z}}\hat{f}(n)\,z^n$ with complex coefficients, such that the norm

$$\|f\|_W=\left(\sum_{n\in\mathbf{Z}}|\hat{f}(n)|^2\,W^2(n)\right)^{1/2}$$

is finite. Condition (4.2) implies that the operator of (formal) multiplication by z on $L^2(W)$ is bounded and has a bounded inverse. We shall denote this operator by T_W. It is unitarily equivalent to the operator $U(\beta)$ with

$$\beta(n)=\frac{W(n+1)}{W(n)},\qquad n\in\mathbf{Z},$$

and every invertible bilateral weighted shift $U(\beta)$ is unitarily equivalent to T_W, with $W=W_\beta$ (see, e.g., [26, Proposition 7]).

For a formal Laurent series f, we shall denote by $\tilde{f}$ the formal Laurent series $\tilde{f}(z)=\sum_{n\in\mathbf{Z}}\hat{f}(-n)\,z^n$, and consider the spaces

$$L^2_o(W)=\{f\in L^2(W)\colon \tilde{f}=-f\},\qquad L^2_e(W)=\{f\in L^2(W)\colon \tilde{f}=f\}.$$

It is clear that these are closed subspaces of $L^2(W)$.

In the sequel we shall assume that W is an even sequence. It is easy to see that in this case, the subspaces $L^2_o(W)$ and $L^2_e(W)$ are orthogonal and $L^2(W)=L^2_o(W)\oplus L^2_e(W)$. Each of these two subspaces is invariant under the operator $T_W+T_W^{-1}$, that is the operator of formal multiplication by $z+z^{-1}$. We shall denote the restriction of this operator to $L^2_o(W)$ by C_W.

We claim that C_W is unitarily equivalent to the operator B_ω on $H^2(\omega)$ (defined in Section 2) with

$$\omega(n)=\frac{W(n+1)}{W(1)},\qquad n\in\mathbf{Z}_+.$$

Condition (4.2) implies that ω satisfies condition (2.2), and it is easy to verify that the transformation $V_W\colon L^2_o(W)\to H^2(\omega)$ defined by

$$(V_Wf)(z)=\sqrt{2}\,W(1)\sum_{n=0}^{\infty}\hat{f}(n+1)\,z^n,\qquad z\in D_\omega,\quad f\in L^2_o(W)$$

is a surjective isometry, with inverse given by

$$(V_W^{-1} g)(z) = \frac{1}{\sqrt{2}\, W(1)} (zg(z) - z^{-1}\tilde{g}(z)), \qquad g \in H^2(\omega).$$

(We regard here the elements of $H^2(\omega)$ as formal Laurent series whose coefficients with negative indices vanish.) A simple computation shows that

$$B_\omega = V_W C_W V_W^{-1} \tag{4.3}$$

and the claim is proved.

Conversely, given a sequence ω of positive numbers on $\mathbf{Z}_+$ which satisfies (2.2), then the sequence W on $\mathbf{Z}$ defined by $W(0) = 1$, and

$$W(n) = \omega(|n| - 1), \qquad n \in \mathbf{Z} \setminus \{0\}$$

satisfies (4.2), and the corresponding operator C_W satisfies (4.3).

Thus every operator C_W is unitarily equivalent to a weighted bi-shift, and conversely, every weighted bi-shift is unitarily equivalent to an operator of that form.

Assume now that M is a proper subspace of $L_o^2(W)$ which is invariant under C_W. We claim that $N = \bigvee_{n \in \mathbf{Z}} T_W^n M$ (the closed linear span in $L^2(W)$ of the spaces $T_W^n M$, $n \in \mathbf{Z}$) is a proper hyperinvariant subspace of T_W. Since N is not the zero space and is invariant under T_W and T_W^{-1}, we only have to show that $N \neq L^2(W)$ (recall that a common invariant subspace of an invertible bilateral weighted shift and its inverse, is a hyper-invariant subspace of that operator). Consider the orthogonal projection P_W of $L^2(W)$ onto $L_o^2(W)$. It is given by

$$P_W f = \frac{f - \tilde{f}}{2}, \qquad f \in L^2(W).$$

A computation shows that for every $n \in \mathbf{Z}$,

$$P_W T_W^n P_W = \tfrac{1}{2}(T_W^n + T_W^{-n})\, P_W,$$

and it follows by induction that $T_W^n + T_W^{-n}$ is a polynomial in $T_W + T_W^{-1}$. Since $M \in \operatorname{Lat} C_W$, these facts imply that

$$P_W T_W^n M \subset M, \qquad \forall n \in \mathbf{Z},$$

and therefore $P_W N \subset M$. But $P_W L^2(W) = L^2_o(W)$ and by assumption $M \neq L^2_o(W)$. Consequently, $N \neq L^2(W)$.

Thus we see that if every weighted bi-shift has a proper invariant subspace, then for every even W, the operator T_W has a hyperinvariant subspace.

Problem 4. Does every operator T_W, when W is even, have a proper hyperinvariant subspace?

Problem 5. Does any of the operators T_W, where W is even, and the sequence

$$\beta(n) = \frac{W(n+1)}{W(n)}, \qquad n \in \mathbf{Z},$$

satisfies conditions (2.9) and (2.11)–(2.13), have a proper hyperinvariant subspace?

By the previous observation, a negative answer to one of these problems provides a negative answer to the invariant subspace problem on Hilbert spaces.

The representation of a bi-shift as a part of an operator of the form $T_W + T_W^{-1}$, can be used to determine all the invariant subspaces of the Toeplitz operator $A = S + S^*$ on the Hardy space $H^2(\mathbf{T})$. (This operator was mentioned by Halmos [17, p. 73] to illustrate the fact that the invariant subspaces for Toeplitz operators are not plainly visible even in the hermitian case.) Following the steps of the previous representation, we obtain that A can be identified with the operator of multiplication by the function $2\cos\theta$ on the subspace of $L^2[-\pi, \pi]$ which consists of all essentially odd functions. From this it is easy to obtain the following description of Lat A: for every measurable subset E of $[\pi, \pi]$, let

$$N_E = \{ f \in H^2(\mathbf{T}) : e^{i\theta} f(e^{i\theta}) = e^{-i\theta} f(e^{-i\theta}) \text{ a.e. on } E \}.$$

Then Lat A is the collection of all these subspaces.

This representation can also be used to determine the spectrum of an arbitrary bi-shift $A(\alpha)$. It is either the interval $[-2, 2]$, or a solid ellipse. More precisely, one can show that if

$$\delta(\alpha) = \lim_{n \to \infty} \left(\sup_{k \in \mathbf{Z}} \frac{\omega_\alpha(|n+k|)}{\omega_\alpha(|k|)} \right)^{1/2}$$

(the limit always exists), then if $\delta(\alpha)=1$, the spectrum of $A(\alpha)$ is $[-2,2]$ (this extends Proposition 3.3), and if $\delta(\alpha)\neq 1$, then the spectrum of $A(\alpha)$ is the ellipse

$$\sigma(A(\alpha))=\left\{z=x+iy:\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\},$$

where $a=\delta(\alpha)+\delta^{-1}(\alpha)$, and $b=|\delta(\alpha)-\delta^{-1}(\alpha)|$.

Using the unitary equivalence established in Section 2, one can also determine the point spectrum $\sigma_{\mathrm{p}}(A(\alpha))$ of $A(\alpha)$.

When $\omega_\alpha\notin l^2(\mathbf{Z}_+)$ it is empty, and when $\omega_\alpha\in l^2(\mathbf{Z}_+)$ it is not empty, and can be described in terms of the quantity

$$r=\limsup_{n\to\infty}(\omega_\alpha(n))^{1/n}$$

(which by the assumptions is in $(0,1]$) and the sequences $s=\{n\omega_\alpha(n)\}_{n\in\mathbf{Z}_+}$ and $q=\{r^{-n}\omega_\alpha(n)\}_{n\in\mathbf{Z}_+}$.

If $r=1$, then $\sigma_p(A(\alpha))=(-2,2)$ when $s\notin l^2(\mathbf{Z}_+)$, and $\sigma_p(A(\alpha))=[-2,2]$ when $s\in l^2(\mathbf{Z}_+)$.

If $r<1$, let $c=r+r^{-1}$, $d=r^{-1}-r$, and consider the ellipse

$$E=\left\{z=x+iy:\frac{x^2}{c^2}+\frac{y^2}{d^2}<1\right\}$$

Then $\sigma_p(A(\alpha))=E$ when $q\notin l^2(\mathbf{Z}_+)$, and $\sigma_p(A(\alpha))=\bar{E}$ when $q\in l^2(\mathbf{Z}_+)$.

It is known that every operator that commutes with invertible bilateral shift T, is the limit in the strong topology of a sequence of Laurent polynomials in T [26, Corollary (b), p. 91]. It is conceivable that an analogous result is true for the operators C_W, hence for weighted bi-shifts.

Conjecture. Every operator that commutes with a weighted bi-shift $A(\alpha)$ is the limit, in the strong operator topology, of a sequence of polynomials in $A(\alpha)$.

APPENDIX: EXAMPLES OF WEIGHTS

We give some examples of sequences which satisfy the various conditions in Section 2. A large number of such examples can be obtained from the following observations.

Assume that φ is a real-valued continuous function on $\mathbf{R}$, with continuous second derivative outside some bounded neighborhood of zero, which satisfies the conditions

$$\varphi'(x) = o(1), \qquad x \to \pm\infty, \tag{A1}$$

and

$$\varphi''(x) = O(x^{-1}), \qquad x \to \pm\infty. \tag{A2}$$

Condition (A2) implies that

$$\varphi(x+1) + \varphi(x-1) - 2\varphi(x) = O(x^{-1}), \qquad x \to \pm\infty,$$

and condition (A1) implies that

$$\varphi(x+1) - \varphi(x) = o(1), \qquad x \to \pm\infty,$$

and that

$$\int_0^R \frac{\varphi(x) + \varphi(-x)}{x^2+1}\, dx = \sum_{j=0}^{[R]} \frac{\varphi(j) + \varphi(-j)}{j^2+1} + O(1), \qquad R \to \infty.$$

([ ] denotes here the integer part.) Therefore, if

$$\limsup_{R\to\infty} \int_0^R \frac{\varphi(x)+\varphi(-x)}{x^2+1}\, dx = \infty \tag{A3}$$

and

$$\liminf_{R\to\infty} \int_0^R \frac{\varphi(x)+\varphi(-x)}{x^2+1}\, dx = -\infty, \tag{A4}$$

then the sequence $p = \varphi|_{\mathbf{Z}}$ satisfies conditions (2.18)–(2.20), and the sequence

$$\beta(n) = \exp(\varphi(n+1) - \varphi(n)), \qquad n \in \mathbf{Z},$$

satisfies conditions (2.9) and (2.11)–(2.13). If, in addition, φ is an even function, then the sequence

$$\alpha(n) = \exp(\varphi(n+1) - \varphi(n)), \qquad n \in \mathbf{Z}_+,$$

satisfies conditions (2.4)–(2.8).

To see some concrete examples, denote

$$u(x) = \frac{|x|}{\log(|x|+e)}, \qquad v(x) = \sin(\log\log\log(|x|+e^3)),$$

and consider the functions φ_1, φ_2 and φ_3 on $\mathbf{R}$ defined by $\varphi_1(x) = u(x)v(x)$, $\varphi_2(x) = u(x)v(x)(1+\operatorname{sign}(x))$ and $\varphi_3(x) = u(x)(v(x)+2\operatorname{sign}(x))$. It is readily verified that these functions satisfy conditions (A1) and (A2), and since for $j = 1, 2, 3$

$$\int_0^R \frac{\varphi_j(x)+\varphi_j(-x)}{x^2+1}\,dx$$

$$= 2\int_0^R \frac{u(x)\,v(x)}{x^2+1}\,dx$$

$$= \sqrt{2}\log\log R \sin\left(\log\log\log R - \frac{\pi}{4}\right) + O(1), \qquad R \to \infty,$$

they satisfy also conditions (A3) and (A4).

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